Journal of Global Optimization (2005) 32: 437–449
DOI 10.1007/s10898-003-2682-3

Generalized Vector Quasi-Variational Inequality Problems Over Product Sets

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(Received 15 August 2003; accepted 19 August 2003)

Abstract. In this paper we consider vector quasi-variational inequality problems over product sets (in short, VQVIP). Moreover we study generalizations of this model, namely problems of a system of vector quasi-variational inequalities (in short, SVQVIP), generalized vector quasi-variational inequality problems over product sets (in short, GVQVIP) and problems of a system of generalized vector quasi-variational inequalities (in short, SGVQVIP). We show that every solution of (VQVIP) (respectively, (GVQVIP)) is a solution of (SVQVIP) (respectively, (SGVQVIP)). By defining relatively pseudomonotone and relatively maximal pseudomonotone maps and by employing a known fixed point theorem, we establish the existence of a solution of (VQVIP) and (SVQVIP). These existence results are then used to derive the existence of a solution of (GVQVIP) and (SGVQVIP), respectively, The results of this paper extend recent results in the literature. They are obtained in a more general setting.

Key words: Generalized vector quasi-variational inequalities, relatively maximal pseudomonotone maps, relatively pseudomonotone maps, systems of generalized vector quasi-variational inequalities, systems of vector quasi-variational inequalities, vector quasi-variational inequalities.

1. Introduction and Model Formulation

In recent years problems of a system of variational inequalities have been studied by several researchers. Such models are related to traffic equilibrium problems, spatial equilibrium problems, Nash equilibrium problems and general equilibrium programming problems. See for example [4–7, 9, 13, 16, 17] and references therein.

To use variational inequalities in the solution of Debreu type equilibrium problems [10], problems of a system of quasi-variational inequalities have been studied in [3, 21, 22] for example.

Inspired by the study of vector variational inequalities by Giannessi [12], systems of vector variational inequalities with their applications have been investigated in [1, 4, 5] and in references therein.

Very recently variational inequalities over product sets have been studied which involve relatively (generalized) monotone maps; see [1, 2, 13] and references therein.

In this paper we consider vector quasi-variational inequality problems over product sets (in short, VQVIP) and problems of a system of VQVIP (in short, SVQVIP). We show that every solution of (VQVIP) is a solution of (SVQVIP). We define the concept of a relatively maximal pseudomonotone map and prove the existence of a solution of (VQVIP) and (SVQVIP) for these maps. As a consequence we derive an existence result for a solution of (VQVIP) and (SVQVIP) for relatively pseudomonotone hemicontinuous maps. We also consider generalized vector quasi-variational inequality problems over product sets (in short, GVQVIP) and problems of a system of generalized vector quasi-variational inequalities (in short, SGVQVIP), that is (VQVIP) and (SVQVIP) for multivalued maps, respectively. By adopting the technique by Yang and Yao [20] we derive the existence of a solution of (GVQVIP) and (SGVQVIP) by using the existence results for a solution of (VQVIP) and (SVQVIP), respectively. The results of this paper extend recent results in the literature. They are derived in a more general setting than before.

Let $I=\{1,2,\dots,m\}$ be a finite index set. For each $i\in I$, let X_i be a real topological vector space and K_i a nonempty convex subset of X_i. Set

$$X=\prod_{i\in I} X_i \quad \text{and} \quad K=\prod_{i\in I} K_i \tag{1}$$

so that for each $x\in X$ we have $x=(x_I : i\in I)$ where $x_i\in X_i$. Let Y be a real topological vector space with a partial order induced by a proper, closed and convex cone with $\operatorname{int} C\neq\emptyset$ where $\operatorname{int} C$ denotes the topological interior of C in Y. For each $i\in I$ let $f_i\colon K\to L(X_i,Y)$ be a map and define $f(x)=(f_i(x))_{i\in I}$ for all $x\in K$, where $L(X_i,Y)$ denotes the space of all continuous linear functions from X_i to Y. For each $i\in I$, let $A_i\colon K\to 2^{K_i}$ be a multivalued map with nonempty convex values. We define a multivalued map $A\colon K\to 2^K$ by $A(x)=\prod_{i\in I} A_i(x)$ for all $x\in K$ where 2^k denotes the family of all subsets of K. We consider the following *vector quasi-variational inequality problem* over the product set K:

$$\text{(VQVIP)}\begin{cases}\text{find}\bar{x}\in K \text{ such that } \bar{x}\in A(\bar{x}) \text{ and}\\ \sum\limits_{i\in I}\langle f_i(\bar{x}), y_i-\bar{x}_i\rangle\notin -\operatorname{int} C, \quad \forall y_i\in A_i(\bar{x}),\ i\in I.\end{cases} \tag{2}$$

The following problem can be termed a *Minty type vector quasi-variational inequality problem* over the product set K:

$$\text{(MVQVIP)}\begin{cases}\text{find } \bar{x}\in K \text{ such that } \bar{x}\in A(\bar{x}) \quad \text{and}\\ \sum\limits_{i\in I}\langle f_i(y), y_i-\bar{x}_i\rangle\notin -\operatorname{int} C, \quad \forall y_i\in A_i(\bar{x}), i\in I.\end{cases} \tag{3}$$

We also consider the following *problem of a system of vector quasi-variational inequalities*:

$$\text{(SVQVIP)}\begin{cases}\text{find } \bar{x}\in K \text{ such that } \bar{x}\in A(\bar{x}) \quad \text{and}\\ \langle f_i(\bar{x}), y_i-\bar{x}_i\rangle \notin -\operatorname{int} C, \quad \forall y_i\in A_i(\bar{x}), i\in I.\end{cases} \tag{4}$$

The (SVQVIP) can be used as a tool to prove the existence of a solution of Debreu type equilibrium problems for vector-valued functions.

If for each $i\in I, A_i(x)=K_i$ for all $x\in K$, then (SVQVIP) reduces to the *problem of a system of vector variational inequalities* (in short, SVVIP). Existence results for a solution of (SVVIP) are established in [4]. As an application of these results, the existence of a solution of Nash equilibrium problems for vector-valued functions [21] is also derived. Below the solution sets of (VQVIP), (MVQVIP) and (SVQVIP) will be denoted by K_s, K_s^m and K_{ss}, respectively.

In case f_i $(i\in I)$ is a multivalued map, (VQVIP) and (SVQVIP) are called *generalized vector quasi-variational inequality problems* over product sets and *problems of a system of generalized vector quasi-variational inequalities*, respectively. More precisely, for each $i\in I$ let $F_i\colon K\to 2^{L(X_i,Y)}$ be a multivalued map with nonempty values and define $F(x)=(F_i(x))_{i\in I}$ for all $x\in K$. We consider the following problems:

$$\text{(GVQVIP)}\begin{cases}\text{find } \bar{x}\in K \text{ and } \bar{u}\in F(\bar{x}) \text{ such that } \bar{x}\in A(\bar{x}) \text{ and}\\ \sum_{i\in I}\langle \bar{u}_i, y_i-\bar{x}_i\rangle \notin -\operatorname{int} C, \quad \forall y_i\in A_i(\bar{x}), i\in I\end{cases} \tag{5}$$

where u_i is the ith component of u;

$$\text{(SGVQVIP)}\begin{cases}\text{find } \bar{x}\in K \text{ and } \bar{u}\in F(\bar{x}) \text{ such that } \bar{x}\in A(\bar{x}) \text{ and}\\ \langle \bar{u}_i, y_i-\bar{x}_i\rangle \notin -\operatorname{int} C, \quad \forall y_i\in A_i(\bar{x}), i\in I.\end{cases} \tag{6}$$

where u_i is the ith component of u. The solution sets of (GVQVIP) and (SGVQVIP) are denoted by K_s^g and K_{ss}^g, respectively.

2. Preliminaries and Basic Results

DEFINITION 2.1. Let K and X be defined as in (1). A family $\{f_i\}_{i\in I}$ of maps $f_i\colon K\to L(X_i,Y)$ is said to be

(i) *relatively pseudomonotone* if for all $x,y\in K$ we have

$$\sum_{i\in I}\langle f_i(x), y_i-x_i\rangle \notin -\operatorname{int} C \Rightarrow \sum_{i\in I}\langle f_i(y), y_i-x_i\rangle \notin -\operatorname{int} C;$$

(ii) *relatively maximal pseudomonotone* if it is relatively pseudomonotone and for all $x, y \in K$ we have

$$\sum_{i \in I} \langle f_i(z), z_i - x_i \rangle \notin -\operatorname{int} C \quad \forall z \in]x, y] \Rightarrow \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle \notin -\operatorname{int} C$$

where z_i is the ith component of z, $]x, y] = \prod_{i \in I}]x_i, y_i]$ and $]x_i, y_i]$ denotes the line segment joining x_i and y_i but not containing x_i.

DEFINITION 2.2. A family $\{f_i\}_{i \in I}$ of maps $f_i : K \to L(X_i, Y)$ is said to be *hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0, 1]$ the mapping $\lambda \mapsto \sum_{i \in I} \langle f_i(x + \lambda z), z_i \rangle$ with $z_i = y_i - x_i$ is continuous, where z_i is the ith component of z.

PROPOSITION 2.1. *If the family $\{f_i\}_{i \in I}$ of maps $f_i : K \to L(X_i, Y)$ is hemicontinuous and relatively pseudomonotone, then it is relatively maximal pseudomonotone.*

Proof. Let $\{f_i\}_{i \in I}$ be hemicontinuous. Assume that for all $x, y \in K$

$$\sum_{i \in I} \langle f_i(z), z_i - x_i \rangle \notin -\operatorname{int} C, \quad \forall z \in]x, y]$$

where z_i is the ith component of z. Since $Y \setminus (-\operatorname{int} C)$ is a cone, we have

$$\sum_{i \in I} \langle f_i(x + \lambda(y - x)), y_i - x_i \rangle \notin -\operatorname{int} C, \quad \forall \lambda \in (0, 1].$$

By hemicontinuity of $\{f_i\}_{i \in I}$ we have

$$\sum_{i \in I} \langle f_i(x), y_i - x_i \rangle \notin -\operatorname{int} C.$$

Hence $\{f_i\}_{i \in I}$ is relatively maximal pseudomonotone. □

The following example shows that the converse of the above lemma is not true in general.

EXAMPLE 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0. \end{cases}$$

Then f is relatively maximal pseudomonotone, but not hemicontinuous.

LEMMA 2.1. *Every solution of* (VQVIP) *is a solution of* (SVQVIP)*, that is* $K_s \subseteq K_{ss}$.

Proof. Let $\bar{x} \in K$ be a solution of (VQVIP). Take arbitrary points $x_i \in A_i(\bar{x})$ for all $i \in I$. Then $x = (x_i : i \in I) \in A(\bar{x})$. Clearly (3) holds for any point y which is defined by $y_i = x_i$ for an arbitrarily fixed $i \in I$ and $y_j = \bar{x}_j$ for all $j \neq i$ since $y \in A(\bar{x})$. Now substituting y in (3) yields (5) after taking i to be $1, 2, \ldots, m$ sequentially. Thus $\bar{x} \in K$ is a solution of (SVQVIP). □

LEMMA 2.2. *If the family* $\{f_i\}_{i\in I}$ *of maps* $f_i : K \to L(X_i, Y)$ *is relatively maximal pseudomonotone and for each* $i \in I$, A_i *is nonempty and convex-valued, then* $K_s = K_s^m$.

Proof. Let $\bar{x} \in K$ be a solution of (MVQVIP). Then $\bar{x} \in A(\bar{x})$ and

$$\sum_{i\in I} \langle f_i(y), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \quad \forall y_i \in A_i(\bar{x}), i \in I.$$

Since for each $i \in I$, $A_i(\bar{x})$ is convex, we have $]\bar{x}_i, y_i] \subset A_i(\bar{x}), \forall i \in I$. Therefore

$$\sum_{i\in I} \langle f_i(z), z_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \quad \forall z_i \in]\bar{x}_i, y_i], i \in I.$$

By relatively maximal pseudomonotonicity of $\{f_i\}_{i\in I}$ we have

$$\sum_{i\in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \quad \forall y_i \in A(\bar{x}), i \in I.$$

Hence $\bar{x} \in K$ is a solution of (VQVIP).

By relatively pseudomonotonicity of $\{f_i\}_{i\in I}$ we have $K_s \subseteq K_s^m$. Hence $K_s = K_s^m$. □

From Proposition 2.1 and Lemma 2.2 we have the following result.

LEMMA 2.3. *If the family* $\{f_i\}_{i\in I}$ *of maps* $f_i : K \to L(X_i, Y)$ *is hemicontinuous and relatively pseudomonotone and for each* $i \in I$, A_i *is nonempty and convex-valued then* $K_s = K_s^m$.

DEFINITION 2.3 (11). Let E be a topological space. A subset D of E is said to be *compactly open* (respectively, *compactly closed*) in E if for any nonempty compact subset L of E, $D \cap L$ is open (respectively, closed) in L.

REMARK 2.1. (a) It is clear from the above definition that every open (respectively, closed) set is compactly open (respectively, compactly closed).

(b) The union or intersection of two compactly open (respectively, compactly closed) sets is compactly open (respectively, compactly closed).

We shall use the,following fixed point theorem due to Chowdhury and Tan[8].

THEOREM 2.1. *Let K be a nonempty convex subset of a topological vector space X (not necessarily Hausdorff) and $S, T: K \to 2^K$ multivalued maps. Assume that the following conditions hold:*

(i) *For all $x \in K$, $S(x) \subseteq T(x)$.*
(ii) *For all $x \in K$, $S(x) \neq \emptyset$.*
(iii) *For all $x \in K$, $T(x)$ is convex.*
(iv) *For all $y \in K$, $S^{-1}(y) := \{x \in K : y \in S(x)\}$ is compactly open.*
(v) *There exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that $K \backslash D \subset S^{-1}(\tilde{y})$.*

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.

3. Existence of Solutions of (VQVIP) and (SVQVIP)

Throughout the remainder of the paper, unless otherwise specified, we assume that for each $i \in I$, $A_i: K \to 2^{k_i}$ is a multivalued map with nonempty and convex values and for all $y_i \in K_i$, $A_i^{-1}(y_i)$ is compactly open in K. We define a multivalued map $A: K \to 2^K$ by $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$ such that the set $\mathcal{F} = \{x \in K : x \in A(x)\}$ is compactly closed.

THEOREM 3.1. *For each $i \in I$ let X_i be a real topological vector space, Y and C be the same as defined above, K_i a nonempty convex subset of X_i, $K = \prod_{i \in I} K_i$ and $\{f_i\}_{i \in I}$ a relatively maximal pseudomonotone family of maps. Assume that there exist a nonempty, closed and compact set D of K and $\bar{y} \in D$ such that $\sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle \in -\text{int}\, C$ for all $x \in K \backslash D$ with $\tilde{y} \in A(x)$. Then the solution set K_s of (VQVIP) is nonempty. Furthermore (SVQVIP) has a solution.*

Proof. For each $x \in K$ define two multivalued maps $P, Q: K \to 2^K$ by

$$P(x) = \left\{ y \in K : \sum_{i \in I} \langle f_i(y), y_i - x_i \rangle \in -\text{int}\, C \right\}$$

and

$$Q(x) = \left\{ y \in K : \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle \in -\text{int}\, C \right\}.$$

Then clearly for each $x \in K$, $Q(x)$ is convex. By relative pseudomonotonicity of $\{f_i\}_{i\in I}$ we have $P(x) \subseteq Q(x)$ for all $x \in K$.

For each $y \in K$ the complement of $P^{-1}(y)$ in K is

$$[P^{-1}(y)]^c = \left\{ x \in K : \sum_{i\in I} \langle f_i(y), y_i - x_i \rangle \notin -\operatorname{int} C \right\}$$

is closed in K and hence $P^{-1}(y)$ is open in K. Therefore $P^{-1}(y)$ is compactly open.

Since for each $i \in I$ and for all $x \in K$, $A_i(x)$ is nonempty and convex, we have $A(x) = \prod_{i\in I} A_i(x)$ is nonempty and convex. Also, since for all $y_i \in K_i$, $A^{-1}(y) = \bigcap_{i\in I} A_i^{-1}(y_i)$ and $A_i^{-1}(y_i)$ is compactly open for each $i \in I$ and for all $y_i \in K_i$, we have $A^{-1}(y)$ is compactly open in K for all $y \in K$.

Now we define two other multivalued maps $S, T : K \to 2^K$ by

$$S(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in \mathcal{F} \\ A(x), & \text{if } x \in K \backslash \mathcal{F} \end{cases}$$

and

$$T(x) = \begin{cases} A(x) \cap Q(x), & \text{if } x \in \mathcal{F} \\ A(x), & \text{if } x \in K \backslash \mathcal{F} \end{cases}$$

Then for all $x \in K$, $T(x)$ is convex and $S(x) \subseteq T(x)$.

Since for each $y \in K$, $A^{-1}(y)$, $P^{-1}(y)$ and $K \backslash \mathcal{F}$ are compactly open and for all $y \in K$

$$S^{-1}(y) = (A^{-1}(y) \cap P^{-1}(y)) \cup ((K \backslash \mathcal{F}) \cap A^{-1}(y))$$

(see the proof of Lemma 2.3 in [11]), we have $S^{-1}(y)$ is compactly open. Now assume to be contrary that for each $x \in \mathcal{F}$, $A(x) \cap P(x) \neq \emptyset$. Then for each $x \in K$, $S(x) \neq \emptyset$. Hence all the conditions of Theorem 2.1 are satisfied. Therefore there exists $x^* \in K$ such that $x^* \in T(x^*)$. From the definition of $\mathcal{F}$ and T we have $\{x \in K : x \in T(x)\} \subseteq \mathcal{F}$. Therefore $x^* \in \mathcal{F}$ and $x^* \in A(x^*) \cap Q(x^*)$ and in particular $\sum_{i\in I} \langle f_i(x^*), x_i^* - x_i^* \rangle = 0 \in -\operatorname{int} C$, a contradiction. Hence there exists $\bar{x} \in \mathcal{F}$ such that $A(\bar{x}) \cap P(\bar{x}) = \emptyset$, that is $\bar{x} \in A(\bar{x})$ and

$$\sum_{i\in I} \langle f_i(y), y_i - x_i \rangle \notin -\operatorname{int} C, \quad \forall y_i \in A_i(\bar{x}), i \in I.$$

By Lemma 2.2 $\bar{x} \in K_s$. It follows from Lemma 2.1 that $\bar{x} \in K$ is a solution of (SVQVIP). □

REMARK 3.1. In Theorem 3.1 we have not assumed any kind of continuity condition.

COROLLARY 3.1. *For each $i \in I$ let X_i be a real topological vector space, Y and C be the same as defined above, K_i a nonempty convex subset of X_i, $K = \prod_{i\in I} K_i$ and $\{f_i\}_{i\in I}$ a hemicontinuous and relatively pseudomonotone family of maps. Assume that there exist a nonempty, closed and compact set D of K and $\tilde{y} \in D$ such that $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle \in -\text{int}C$ for $x \in K\setminus D$ with $\tilde{y} \in A(x)$. Then the solution set K_s of $(VQVIP)$ is nonempty. Furthermore $(SVQVIP)$ has a solution.*

4. Existence of Solutions of (GVQVJP) and (SGVQVIP)

In this section we adopt the technique of Yang and Yao [20] to derive existence results for a solution of (GVQVIP) and (SGVQVIP) with help of the results of Section 3.

LEMMA 4.1. *Every solution of* (GVQVIP) *is a solution of* (SGVQVIP)*, that is $K_s^g \subseteq K_{ss}^g$.*

DEFINITION 4.1. Let K and X be as defined in (1). A family $\{F_i\}_{i\in I}$ of multivalued maps $F_i\colon K \to 2^{L(X_i,Y)}$ is said to be

(i) *relatively pseudomonotone* if for all $x, y \in K$ and for all $u \in F(x), v \in F(y)$ we have

$$\sum_{i\in I}\langle u_i, y_i - x_i\rangle \notin -\text{int}\,C \Rightarrow \sum_{i\in I}\langle v_i, y_i - x_i\rangle \notin -\text{int}\,C$$

where u_i is the ith component of u;

(ii) *relatively maximal pseudomonotane* if it is relatively pseudomonotone and for all $x, y \in K$ and for all $u \in F(x)$ we have

$$\sum_{i\in I}\langle w_i, z_i - x_i\rangle \notin -\text{int}\,C \quad \forall w_i \in F_i(z), i \in I \text{ and } z \in]x, y]$$
$$\Rightarrow \sum_{i\in I}\langle u_i, y_i - x_i\rangle \notin -\text{int}\,C$$

where z_i, is the ith component of z;

(iii) *u-hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0, 1]$ the mapping $\lambda \mapsto \sum_{i\in I}\langle F_i(x + \lambda z), z_i\rangle$ with $z = y - x$ is upper semicontinuous at 0, where z_i is the ith component of z.

LEMMA 4.2. *If $\{F_i\}_{i\in I}$ is u-hemicontinuous and relatively pseudomanotone, then it is relatively maximal pseudomonotone.*

Proof. Suppose that for all $x, y \in K$ and for all $u \in F(x)$ we have

$$\sum_{i \in I} \langle u_i, y_i - x_i \rangle \in -\operatorname{int} C.$$

Set $z = ty + (1-t)x$ for $0 < t \leqslant 1$, that is $z \in]x, y]$. Then by u-hemicontinuity of $\{F_i\}_{i \in I}$, there exists a $\delta > 0$ such that

$$\sum_{i \in I} \langle w_i, y_i - x_i \rangle \in -\operatorname{int} C, \quad \forall w \in F_{(z)}, \text{and } t \in (0, \delta).$$

Since $t(y_i - x_i) = z_i - x_i$ for each $i \in I$, we have

$$\sum_{i \in I} \langle w_i, z_i - x_i \rangle \in -\operatorname{int} C, \quad \forall w_i \in F_i(z), i \in I \text{ and } z \in]x, y].$$

This completes the proof. □

Let W and Z be topological vector spaces and U a subset of W. Let G: $U \to 2^{L(W,Z)}$ and $g: U \to L(W, Z)$. Recall that g is a *selection* of G on U if $g(x) \in G(x)$ for all $x \in U$. Furthermore the function g is called a *continuous selection* of G on U if it is continuous on U and a selection of G on U.

For results on the existence of a continuous selection we refer to [15, 19, 18] and references therein.

LEMMA 4.3. *For each* $\in I$ *if* f_i *is a selection of* F_i *on* K *and* $\bar{x} \in K$ *is a solution of* (VQVIP)*, then* $(\bar{x}, \bar{u})$ *is a solution of* (GVQVIP) *with* $\bar{u}_i \in f_i(\bar{x})$ *for all* $i \in I$.

Proof. Assume that $\bar{x} \in K$ is a solution of (VQVIP). Then $\bar{x} \in A(\bar{x})$ and

$$\sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \quad \forall y_i \in A_i(\bar{x}), i \in I.$$

Let $\bar{u}_i = f_i$ so that $\bar{u} = f(\bar{x})$. For each $i \in I$ since f_i is a selection of F_i, we have $\bar{u} \in F(\bar{x})$ such that $\bar{x} \in A(\bar{x})$ and

$$\sum_{i \in I} \langle \bar{u}_i, y_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \quad \forall y_i \in A_i(\bar{x}), i \in I.$$

Hence $(\bar{x}, \bar{u})$ is a solution of (GVQVIP). □

LEMMA 4.4. *For each* $i \in I$ *let* $f_i: K \to L(X_i, Y)$ *be a selection of a multivalued map* $F_i: K \to 2^{L(X_i,Y)}$ *on K. If the family* $\{F_i\}_{i \in I}$ *of multivalued maps is relatively maximal pseudomonotone, then* $\{f_i\}_{i \in I}$ *is also relatively maximal pseudomonotone.*

THEOREM 4.1. *For each $i \in I$, let X_i be a real topological vector space, Y and C be the same as defined above, K_i nonempty convex subset of X_i and $K = \prod_{i\in I} K_i$. Further assume that*

(i) *$\{F_i\}_{i\in I}$ is a relatively maximal pseudomonotone multivalued map;*
(ii) *for each $i \in I$ there exists a selection (not necessarily continuous) f_i of F_i on K;*
(iii) *there exist a nonempty, closed and compact set D of K and $\tilde{y} \in D$ such that $\sum_{i\in I} \langle f_i(x), \tilde{y}_i - \bar{x}_i \rangle \in -\text{int}\, C \quad for\ all\ x \in K \backslash D\ with\ \tilde{y} \in A(x).$*

Then the solution set K_s^g of (GVQVIP) *is nonempty. Furthermore* (SGVQ-VIP) *has a solution.*

Proof. By assumption (ii) for each $i \in I$ there is a function f_i such that $f_i(x) \in F_i(x)$ for all $x \in K$. Lemma 4.4 implies that $\{f_i\}_{i\in I}$ is relatively maximal pseudomonotone. Then all the conditions of Theorem 3.1 are satisfied and therefore there exists a solution $\bar{x} \in K$ of (VQVIP). For each $i \in I$ let $\bar{u}_i = f_i(\bar{x})$ so that $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then by Lemma 4.3 $(\bar{x}, \bar{u}) \in K_s^g$, where $\bar{u} = (\bar{u}_i)_{i\in I}$. In view of Lemma 4.1. $(\bar{x}, \bar{u})$ is a solution of (SGVQVIP). □

THEOREM 4.2. *For each $i \in I$ let X_i be a real topological vector space, Y and C be as defined above, K_i a nonempty convex subset of X_i and $K = \prod_{i\in I} K_i$ Furthermore assume that*

(i) *$\{F_i\}_{i\in I}$ is a relatively maximal pseudomonotone multivalued map;*
(ii) *for each $i \in I$ there exists a continuous selection f_i of F_i on K;*
(iii) *there exist a nonempty, closed and compact set D of K and $\tilde{y} \in D$ such that $\sum_{i\in I} \langle f_i(x), \tilde{y}_i - x_i \rangle \in -\text{int}\, C \quad for\ all\ x \in K \backslash D\ with\ \tilde{y} \in A(x).$*

Then the solution set K_s^g of (GVQVIP) *is nonempty. Furthermore* (SGVQ-VIP) *has a solution.*

Proof. It follows from condition (ii) that for each $i \in I$ there is a continuous function f_i such that $f_i(x) \in F_i(x)$ for all $x \in K$. Therefore by Lemma 4.4 $\{f_i\}_{i\in I}$ is relatively pseudomonotone and continuous, and so it is, relatively pseudomonotone and hemicontinuous. Then all conditions of Corollary 3.1 are satisfied, and therefore there exists a solution $\bar{x} \in K$ of (VQVIP). For each $i \in I$ let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then Lemma 4.3 implies that $(\bar{x}, \bar{u}) \in K_s^g$, where $\bar{u} = (\bar{u}_i)_{i\in I}$. In view of Lemma 4.1 $(\bar{x}, \bar{u})$ is a solution of (SGVQVIP). □

Now we provide an example of a relatively maximal pseudomonotone multivalued map which has a selection but not a continuous selection.

EXAMPLE 4.1. Let $F: [0,\infty) \to 2^{R}$ be defined by

$$F(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ [1+x, \infty) & \text{if } x > 0. \end{cases}$$

Then F is relatively maximal pseudomonotone, but F does not have any continuous selection since any selection of F is discontinuous at 0.

REMARK 4.2. If I is a Singleton, then Theorem 4.2 extends Theorem 3.1 [20] (for a fixed cone) to generalized vector quasi-variational inequalities in a more general setting and for a noncompact set.

In view of Lemma 4.2 we have the following result.

COROLLARY 4.1. *For each $i \in I$ let X_i be a real topological vector space, Y and C be as defined above, K_i a nonempty convex subset of X_i and $K = \prod_{i\in I} K_i$. Furthermore assume that*

(i) *$\{F_i\}_{i\in I}$ is u-hemicontinuous and relatively pseudomonotone;*
(ii) *for each $i \in I$ there exists a selection* (*not necessarily continuous*) *f_i of F_i on K;*
(iii) *there exist a nonempty, closed and compact set D of K and $\tilde{y} \in D$ such that $\sum_{i\in I} \langle f_i(x), \tilde{y}_i - \bar{x}_i\rangle \in -\text{int}\, C$ for all $x \in K\backslash D$ with $\tilde{y} \in A(x)$.*

Then the solution set K_s^g of (GVQVIP) *is nonempty. Furthermore* (SGVQ-VIP) *has a solution.*

REMARK 4.3. Note that we did not provide an answer to the following open question posed by Yang and Yao [20]: for each $i \in I$, if F_i is a u-hemicontinuous multivalued map, then under what conditions does there exist a hemicontinuous function f_i such that $f_i(x) \in F_i(x)$ for all $x \in K$? But we did establish the existence of a solution of (GVQVIP) and (SGVQ-VIP) which include generalized vector variational inequality problems as a special case under the relative pseudomonotonicity and u-hemicontinuity assumption.

5. Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions. In this research, the first author was partially supported by the Major Research Project of University Grants Commission, No. F. 8-1/2002 (SR-1), New Delhi, Govt. of India while the third author was supported by the National Science Council of the Republic of China. The first author is also grateful to the Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia for providing excellent research facilities to carry out this work.

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